

# THE UNIT BALL OF AN INJECTIVE OPERATOR SPACE HAS AN EXTREME POINT

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**ABSTRACT.** We define an  $AW^*$ -TRO as an off-diagonal corner of an  $AW^*$ -algebra, and show that the unit ball of an  $AW^*$ -TRO has an extreme point. In particular, the unit ball of an injective operator space has an extreme point, which answers a question raised in [8] affirmatively. We also show that an  $AW^*$ -TRO (respectively, an injective operator space) has an ideal decomposition, that is, it can be decomposed into the direct sum of a left ideal, a right ideal, and a two-sided ideal in an  $AW^*$ -algebra (respectively, an injective  $C^*$ -algebra). In particular, we observe that  $AW^*$ -TRO, hence an injective operator space, has an algebrization which admits a quasi-identity.

Recall that an operator space  $X$  is called a *triple system* or a *ternary ring of operators* (TRO for short) if there exists a complete isometry  $\iota$  from  $X$  into a  $C^*$ -algebra such that  $\iota(x)\iota(y)^*\iota(z) \in \iota(X)$  for all  $x, y, z \in X$ . A theorem of Ruan and Hamana (independently) states that an operator space  $X$  is injective if and only if it is an off-diagonal corner of an injective  $C^*$ -algebra, i.e., there exist an injective  $C^*$ -algebra  $\mathcal{A}$  and projections  $p, q \in \mathcal{A}$  (meaning  $p = p^2 = p^*$  and  $q = q^2 = q^*$ ) such that  $X$  is completely isometric to  $p\mathcal{A}q$  (Theorem 4.5 in [14] and Theorem 3.2 (i) in [2]). In particular, an injective operator space is a TRO. Noting that an injective  $C^*$ -algebra is monotone complete and hence an  $AW^*$ -algebra, the Ruan-Hamana theorem motivates the following definition. (The reader is referred to [15] for a modern account of and recent progress in monotone complete  $C^*$ -algebras and  $AW^*$ -algebras.)

**Definition 1.** We say that an operator space  $X$  is an  $AW^*$ -TRO if there exist an  $AW^*$ -algebra  $\mathcal{A}$  and projections  $p, q \in \mathcal{A}$  such that  $X$  is completely isometric to  $p\mathcal{A}q$ .

**Remark 2.** (1) Our definition of an  $AW^*$ -TRO is weaker than the one given in [12] (Definition 6.2.1) where an  $AW^*$ -TRO is defined as a TRO whose linking  $C^*$ -algebra is an  $AW^*$ -algebra. This condition is so strong that even some injective operator spaces fail to be  $AW^*$ -TROs in this sense. For instance, a countably-infinite-dimensional column Hilbert space is an injective operator space ([13]) and hence a TRO, however, its linking  $C^*$ -algebra is not unital, and so is not an  $AW^*$ -algebra. In our belief, disqualifying an injective operator space, which is an off-diagonal corner of an  $AW^*$ -algebra, from being an  $AW^*$ -TRO is not befitting to its name, so in this paper we use the term “ $AW^*$ -TRO” in the sense of Definition 1 above, and hence

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an injective operator space is an  $AW^*$ -TRO. Also this definition is consistent with that of a  $W^*$ -TRO which is an off-diagonal corner of a  $W^*$ -algebra but its linking  $C^*$ -algebra need not be a  $W^*$ -algebra.

- (2) With this modified definition of  $AW^*$ -algebras and  $\mathcal{L}_T := \begin{bmatrix} p\mathcal{A}p & p\mathcal{A}q \\ q\mathcal{A}p & q\mathcal{A}q \end{bmatrix} \subseteq \mathbb{M}_2(\mathcal{A})$ , where  $\mathcal{A}$ ,  $p$ , and  $q$  are as in Definition 1, all Theorems, Corollaries, and Lemmas in Sections 6.2 and 6.3 of [12] are valid except for Statement 6.2.2 and Corollary 6.2.6 there.

**Theorem 3.** *The unit ball (always assumed to be norm-closed) of an  $AW^*$ -TRO has an extreme point. In particular, the unit ball of an injective operator space has an extreme point, which answers a question raised in [8] (Question 2) affirmatively.*

*Proof.* Let  $X$  be an  $AW^*$ -TRO. We may assume that  $X = p\mathcal{A}q$ , where  $\mathcal{A}$  is an  $AW^*$ -algebra and  $p, q \in \mathcal{A}$  are projections. By the comparison theorem in [3], there exist unique central projections  $r, t, l \in \mathcal{A}$  satisfying  $r + t + l = 1$  such that  $rp \prec rq$ ,  $tp \sim tq$ , and  $lp \succ lq$ . (Here  $rp \prec rq$  means  $rp \preceq rq$  but  $rp \not\sim rq$ , however,  $0 \prec 0$  is allowed.) That is, there exist partial isometries  $u, v, w \in \mathcal{A}$  such that  $uu^* = rp$ ,  $u^*u \leq rq$ ,  $vv^* = tp$ ,  $v^*v = tq$ ,  $ww^* \leq lp$ , and  $w^*w = lq$ . Let  $e := u + v + w (\in p\mathcal{A}q)$ , then it is easy to check that  $(p - ee^*)\mathcal{A}(q - e^*e) = \{0\}$ . Thus by a variation of Kadison's theorem (Theorem 1 in [4]; see Proposition 1.4.8 in [11] or Proposition 1.6.5 in [16] for the variation we need here),  $e$  is an extreme point of the unit ball of  $p\mathcal{A}q$ .  $\square$

From the proof above we obtain “ideal decompositions” for  $AW^*$ -TROs and injective operator spaces similar to the ones done for TROs with predual in [7]. The technique we use here is to embed an off-diagonal corner into the diagonal corners which is a modification of the technique developed in [1] and is employed in [7].

**Corollary 4.** *An  $AW^*$ -TRO (respectively, injective operator space) can be decomposed into the direct sum of TROs  $X_T$ ,  $X_L$ , and  $X_R$ :*

$$X = X_T \overset{\infty}{\oplus} X_L \overset{\infty}{\oplus} X_R$$

*so that there is a complete isometry  $\iota$  from  $X$  into an  $AW^*$ -algebra (respectively, an injective  $C^*$ -algebra) in which  $\iota(X_T)$ ,  $\iota(X_L)$ , and  $\iota(X_R)$  are a two-sided, left, and right ideal, respectively, and*

$$\iota(X) = \iota(X_T) \overset{\infty}{\oplus} \iota(X_L) \overset{\infty}{\oplus} \iota(X_R).$$

*Proof.* Let  $X$  be an  $AW^*$ -TRO, and assume that  $X = p\mathcal{A}q$ , where  $\mathcal{A}$  is an  $AW^*$ -algebra and  $p, q \in \mathcal{A}$  are projections. Let  $r, t, l \in p\mathcal{A}q$  as in the proof of Theorem 3, and put  $X_T := tX$ ,  $X_L := lX$ , and  $X_R := rX$ , then  $X = X_T \overset{\infty}{\oplus} X_L \overset{\infty}{\oplus} X_R$ . Let  $\mathcal{B} := p\mathcal{A}p \overset{\infty}{\oplus} q\mathcal{A}q$  which is an  $AW^*$ -algebra, since  $p\mathcal{A}p$  and  $q\mathcal{A}q$  are so by Theorem 2.4 in [10]. For each  $x \in X$ , let  $x_T := tx$ ,  $x_L := lx$ , and  $x_R := rx$ , and define a mapping  $\iota : X \rightarrow \mathcal{B}$  by  $\iota(x) := (x_T + x_L)e^* \oplus e^*x_R$ , then clearly  $\iota(X) = \iota(X_T) \overset{\infty}{\oplus} \iota(X_L) \overset{\infty}{\oplus} \iota(X_R)$ . We claim that  $\iota$  is a complete isometry.  $\|\iota(x)\| = \max\{\|(x_T + x_L)e^*\|, \|e^*x_R\|\} = \max\{\|(x_T + x_L)e^*e(x_T + x_L)^*\|^{1/2}, \|x_R^*ee^*x_R\|^{1/2}\} = \max\{\|x_T v^*v x_T^* + x_L w^*w x_L^*\|^{1/2}, \|x_R^* u u^* x_R\|^{1/2}\} = \max\{\|x_T x^* + x_L x^*\|^{1/2}, \|x_R^* r x\|^{1/2}\} = \max\{\|(t + l)x\|, \|r x\|\} = \|(t + l + r)x\| = \|x\|$ , which shows that  $\iota$

is an isometry. A similar calculation works at the matrix level, which concludes that  $\iota$  is a complete isometry. Clearly,  $\iota(X_T)$ ,  $\iota(X_L)$ , and  $\iota(X_R)$  are respectively a two-sided, left, and right ideals in  $\mathcal{B}$ , and thus we are done. The proof in the case that  $X$  is an injective operator space is identical noting that  $\mathcal{B}$  is an injective  $C^*$ -algebra in this case.  $\square$

**Remark 5.** (1) In the proof above it is also possible to define  $\iota : X \rightarrow \mathcal{B}$  by  $\iota(x) := x_L e^* \oplus e^*(x_R + x_T)$  for  $x \in X$ .

- (2) A TRO  $X$  with predual can be considered as an off-diagonal corner of a von Neumann algebra  $\mathcal{A}$  (see the beginning of the proof of the Theorem in [7]), thus the above argument gives an alternate and simpler proof of the Theorem in [7] noting that  $\mathcal{B}$  in the above proof is a von Neumann algebra in this case. The simplicity of this alternate proof is attributed to the use of the comparison theorem for projections.

The following corollary is straightforward from the corollary above. The reader is referred to [5], [6], or [9] for quasi-multipliers and algebrizations of operator spaces, and Definition 4.2 (i) in [8] for quasi-identities.

**Corollary 6.** *An  $AW^*$ -TRO, hence an injective operator space, has an algebrization which admits a quasi-identity of norm 1.*

*Proof.* It is straightforward to check that  $(v + w)e^* \oplus e^*u$  serves as a quasi-identity of  $\iota(X)$  in the proof of Corollary 4.  $\square$

**Remark 7.** *An element  $e^* \in X^*$  in the proof of Corollary 4 can be identified as a quasi-multiplier of  $X$  noting that  $X^* \subseteq \mathcal{QM}(X)$  if  $X$  is a TRO, where  $\mathcal{QM}(X)$  is the quasi-multiplier space of  $X$ , and  $\iota$  is the “algebrization” by  $e^*$ .*

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